Dispersion management and the direct scattering transform

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It is shown that the direct scattering transform may be used to derive a perturbative theory of quasiperiodic field propagation in nonlinear fiber lines managed through zero-average, piecewise-constant dispersion maps. In this scheme, the field is propagated exactly and it is the quasiperiodicity property which is approximated. The resulting quasiperiodicity conditions are shown to agree with the dispersion managed nonlinear Schrödinger equation up to order 4.

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I. INTRODUCTION

A great deal of effort has recently been dedicated to the study of field propagation in long-haul fiber lines. In such fibers, the transmission over long distances forces one to treat the nonlinearity and the pulse evolution turns out to be governed by the nonlinear Schrödinger equation (NLS),

$$i\frac{\partial u}{\partial z}-\frac{\beta}{2}\frac{\partial^2 u}{\partial \tau^2}+\kappa|u|^2u=0, \tag{1}$$

where z is the distance along the fiber, τ is the retarded time, β is the dispersion coefficient, κ is the nonlinearity, and $u(z,\tau)$ is the slowly varying field envelope.

In the past few years, dispersion management (DM) has arisen as the most promising technique for minimizing spectral broadening, timing jitter [1–3], and crosstalk [4,5] in such systems. Up to very recently, the analytical description of DM had been achieved mainly through three approaches: an averaging method [6], a regular perturbation theory [7,8], and a multiple scales analysis [9]. These methods express the periodicity condition on the field as a nonlinear integral constraint, known as the DMNLS. However, these formalisms also have the drawback of evolving the field in approximate ways. The present goal is to study DM using an exact field propagation, for the experimentally interesting case of a piecewise-constant dispersion profile. This idea has already been tackled by a few authors [10,11], but the integral periodicity conditions arising from such an approach have never actually been identified.

To this end, a review of the scattering transform will first be presented, the focus being set on aspects which are pertinent to the DM problem, and the main goal being to expose the notation subsequently used; the reader is referred to Refs. [12,13] for more detailed treatments. The direct scattering transform will then be used to devise a perturbation scheme for quasiperiodic field propagation and the result will be compared with the DMNLS description.

II. THE SCATTERING TRANSFORM

A. Definitions

Consider the 2×2 Ablowitz-Kaup-Newell-Segur (AKNS) linear problem for the scattering of a two-component field $\boldsymbol{\chi}(z,\tau)$ by a pair of potentials $q(z,\tau)$ and $v(z,\tau)$

$$\partial_\tau \boldsymbol{\chi}=\begin{pmatrix} -i\lambda & q \\ v & i\lambda \end{pmatrix}\boldsymbol{\chi}, \tag{2a}$$

$$\partial_z \boldsymbol{\chi}=\beta\begin{pmatrix} -i\lambda^2-\frac{i}{2}qv & \lambda q+\frac{i}{2}\partial_\tau q \\ \lambda v-\frac{i}{2}\partial_\tau v & i\lambda^2+\frac{i}{2}qv \end{pmatrix}\boldsymbol{\chi}. \tag{2b}$$

The first step in the study of this linear problem is to consider z fixed and focus on the scattering equation (2a). Assuming the potentials to be localized in τ, it is possible to define four solutions to Eq. (2a) according to their asymptotical behaviors at infinity. Consider then the eigenfunctions $\boldsymbol{\psi}_-$ and $\boldsymbol{\phi}_-$, behaving, respectively, as $\binom{1}{0}e^{-i\lambda\tau}$ and $\binom{0}{-1}e^{i\lambda\tau}$ for $\tau\to-\infty$, as well as the eigenfunctions $\boldsymbol{\phi}_+$ and $\boldsymbol{\psi}_+$, behaving as $\binom{0}{1}e^{i\lambda\tau}$ and $\binom{1}{0}e^{-i\lambda\tau}$ for $\tau\to+\infty$. In terms of these solutions and of the Wronskian, $W(\boldsymbol{\chi},\boldsymbol{\xi})\equiv\chi_1\xi_2-\chi_2\xi_1$, scattering coefficients may be defined according to

$$a(\lambda)=W(\phi_-,\phi_+), \tag{3a}$$

$$\overline{a}(\lambda)=W(\psi_-,\psi_+), \tag{3b}$$

$$b(\lambda)=-W(\phi_-,\psi_+), \tag{3c}$$

$$\overline{b}(\lambda)=W(\psi_-,\phi_+). \tag{3d}$$

Assuming the potentials to vanish faster than any algebraic function

$$\int_{-\infty}^{\infty}d\tau|\tau|^n\begin{pmatrix} r(\tau) \\ q(\tau) \end{pmatrix}<\infty \qquad (\forall \quad n\in\mathbb{N}) \tag{4}$$

implies that $a(\lambda)$, $\overline{a}(\lambda)$ are, respectively, analytic in the upper and lower halves of the complex λ plane, including the

*Electronic address: yxudous@phy.ulaval.ca real axis. It can also be shown that Eq. (4) implies $b(\lambda)$ and $\overline{b}(\lambda)$ to be analytic on the real axis.

Now for generic potentials, the AKNS system will possess a continuous spectrum of unbounded eigenfunctions for $\lambda=k\in\mathbb{R}$ as well as a discrete set of bounded ones for $\lambda=\lambda_k\in\Lambda$ and $\lambda=\overline{\lambda}_k\in\overline{\Lambda}$. Here, Λ and $\overline{\Lambda}$ denote the sets of zeros of $a(\lambda)$ and $\overline{a}(\lambda)$, which reside, respectively, in the upper and lower halves of the complex λ plane. The scattering data of the potentials q and v may then be defined in terms of these spectra as the set

$$\mathcal{S}[q,v]\equiv\{\Lambda,\overline{\Lambda},b(\lambda),\overline{b}(\lambda)\}, \tag{5}$$

where it is understood that $\lambda\in\mathbb{R}\cup\Lambda\cup\overline{\Lambda}$.

A link between these scattering data and the NLS may be established by noticing that for any continuous $\boldsymbol{\chi}$ to solve Eq. (2), the potentials must satisfy the compatibility conditions

$$\partial_z q-\frac{i}{2}\beta\partial_\tau^2 q+i\beta q^2 v=0, \tag{6a}$$

$$\partial_z v+\frac{i}{2}\beta\partial_\tau^2 v-i\beta q v^2=0. \tag{6b}$$

An identification of these with Eq. (1) can thus be achieved by parametrizing the potentials in terms of the NLS field $u(z,\tau)$ according to

$$q(z,\tau)=i\sigma\alpha u^*(z,\tau), \tag{7a}$$

$$v(z,\tau)=i\alpha u(z,\tau), \tag{7b}$$

where $\alpha\equiv\sqrt{|\kappa/\beta|}$ and $\sigma\equiv\mathrm{sgn}(-\kappa/\beta)$.

B. Symmetries and scattering data

It is well known that specific symmetries of the potentials can drastically constrain the scattering data. For example, under the NLS symmetry (7), half of the data become redundant,

$$\overline{\lambda}_k=\lambda_k^*, \tag{8a}$$

$$\overline{b}(\lambda)=\sigma b^*(\lambda^*), \tag{8b}$$

and the scattering transform may be truncated to

$$\mathcal{S}[u]=\{\Lambda,b(\lambda)\}, \tag{9}$$

where $\lambda\in\mathbb{R}\cup\Lambda$. Another symmetry which will be put to good use is that of parity. Following Ref. [10], suppose that $u(z,\tau)=\mathcal{P}u(z,-\tau)$, with $\mathcal{P}=\pm1$. Then the transformed field $\tilde{\boldsymbol{\chi}}$ defined as

$$\begin{pmatrix}\tilde{\chi}_1\\ \tilde{\chi}_2\end{pmatrix}\equiv\begin{pmatrix}\chi_1(-\lambda,-\tau)\\ -\mathcal{P}\chi_2(-\lambda,-\tau)\end{pmatrix} \tag{10}$$

satisfies the same scattering equation as $\boldsymbol{\chi}$ in τ space. Examining the asymptotic behavior of these transformed solutions, one finds that

$$\tilde{\boldsymbol{\phi}}_-(\lambda,\tau)=\boldsymbol{\psi}_+(\lambda,\tau), \tag{11a}$$

$$\tilde{\boldsymbol{\psi}}_-(\lambda,\tau)=\mathcal{P}\boldsymbol{\phi}_+(\lambda,\tau), \tag{11b}$$

$$\tilde{\boldsymbol{\phi}}_+(\lambda,\tau)=\mathcal{P}\boldsymbol{\psi}_-(\lambda,\tau), \tag{11c}$$

$$\tilde{\boldsymbol{\psi}}_+(\lambda,\tau)=\boldsymbol{\phi}_-(\lambda,\tau). \tag{11d}$$

Using these identifications in the Wronskian formulas (3), it can be shown that simultaneous parity and NLS symmetries imply

$$\mathrm{Re}\{\lambda_k\}=0, \tag{12a}$$

$$b(\lambda)=\mathcal{P}b(-\lambda). \tag{12b}$$

The last symmetry constraints that will be used were also obtained in Ref. [10], and they relate two different scattering problems. Consider an unprimed AKNS system S with potential u and a primed system S' with potential $u'=u^*e^{i\theta}$. One can show that if $\boldsymbol{\chi}$ solves S, then the transformed field $\tilde{\boldsymbol{\chi}}$ given by

$$\begin{pmatrix}\tilde{\chi}_1\\ \tilde{\chi}_2\end{pmatrix}\equiv\begin{pmatrix}\chi_2(-\lambda,\tau)\\ \sigma e^{i\theta}\chi_1(-\lambda,\tau)\end{pmatrix} \tag{13}$$

solves S'. The asymptotics then lead to the identifications

$$\tilde{\boldsymbol{\phi}}_-(\lambda,\tau)=-\sigma e^{i\theta}\boldsymbol{\psi}'_-(\lambda,\tau), \tag{14a}$$

$$\tilde{\boldsymbol{\psi}}_-(\lambda,\tau)=-\boldsymbol{\phi}'_-(\lambda,\tau), \tag{14b}$$

$$\tilde{\boldsymbol{\phi}}_+(\lambda,\tau)=\boldsymbol{\psi}'_+(\lambda,\tau), \tag{14c}$$

$$\tilde{\boldsymbol{\psi}}_+(\lambda,\tau)=\sigma e^{i\theta}\boldsymbol{\phi}'_+(\lambda,\tau), \tag{14d}$$

and the Wronskian formulas imply

$$\lambda'_k=-\lambda_k^*, \tag{15a}$$

$$b'(\lambda)=-e^{i\theta}b^*(-\lambda^*). \tag{15b}$$

This result will play an important role in the subsequent analysis.

C. Direct and inverse scattering

The problem of computing the scattering data corresponding to a given potential $u(\tau)$ is referred to as the direct scattering problem. This can be achieved by solving Eq. (2a) in the integral form

$$\zeta_1(\tau,\lambda)=1-\sigma\alpha^2\int_{-\infty}^{\tau}d\tau_1\int_{-\infty}^{\tau_1}d\tau_2 u^*(\tau_1)u(\tau_2)\times e^{2i\lambda(\tau_1-\tau_2)}\zeta_1(\tau_2,\lambda), \tag{16a}$$

$$\zeta_2(\tau,\lambda)=i\alpha\int_{-\infty}^{\tau}d\tau_1 u(\tau_1)e^{-2i\lambda\tau_1} -\sigma\alpha^2\int_{-\infty}^{\tau}d\tau_1\int_{-\infty}^{\tau_1}d\tau_2 u^*(\tau_2)u(\tau_1) \times e^{2i\lambda(\tau_2-\tau_1)}\zeta_2(\tau_2,\lambda) \tag{16b}$$

for the functions defined by $\zeta_1(\tau,\lambda)\equiv(\phi_-)_1 e^{i\lambda\tau}$ and $\zeta_2(\tau,\lambda)\equiv(\phi_-)_2 e^{-i\lambda\tau}$. The evaluation of the Wronskian formulas at infinity then allows one to construct the data through

$$\lim_{\tau\to\infty}\zeta_1(\tau,\lambda_k)=0, \tag{17a}$$

$$b(\lambda)=\lim_{\tau\to\infty}\zeta_2(\tau,\lambda). \tag{17b}$$

Conversely, the problem of constructing a potential $u(\tau)$, given its scattering data, is well defined and is referred to as the inverse scattering problem

$$\mathcal{S}^{-1}[\{\Lambda,b(\lambda)\}]=u(\tau) \quad (\lambda\in\mathbb{R}\cup\Lambda). \tag{18}$$

This may be achieved by solving a linear integral condition known as the Marchenko equation [12,13]. The knowledge of the scattering data is thus strictly equivalent to that of the field. Finally, it should be emphasized that both the direct and inverse problems are considered at a fixed value of z and therefore pertain only to the first half (2a) of the AKNS system. The usefulness of transforming back and forth between the potential and its scattering data is due to the simple evolution of latter in z space

$$\lambda_k(z)=\lambda_k(z_0), \tag{19a}$$

$$b(z,\lambda)=b(z_0,\lambda)e^{2i\beta(z-z_0)\lambda^2}. \tag{19b}$$

One can then solve for the evolution of the NLS field from z_0 to z by transforming to the scattering data at z_0, evolving these to z and then taking the inverse transform to recover the field.

III. THE DM SOLITON

A. Quasiperiodicity and scattering data

Consider now applying the spectral methods of the preceding section to a cascaded fiber-line system with a two-step piecewise-constant dispersion map, where the jth fiber leg ($j=1,2$) is characterized by its length ℓ_j, second-order dispersion β_j'', and nonlinearity γ. The evolution of the pulse in each leg is governed by the NLS

$$i\frac{\partial u_j}{\partial z_j}-\frac{\beta_j}{2}\frac{\partial^2 u_j}{\partial\tau^2}+\kappa_j|u_j|^2u_j=0. \tag{20}$$

Here, the variables and field are dimensionless, z_j being scaled to ℓ_j, τ being scaled to the junction-field width T_B, and $u_j(z_j,\tau)$ being scaled to the junction-field amplitude A_B, with the scaled dispersion and nonlinearity, respectively, given by

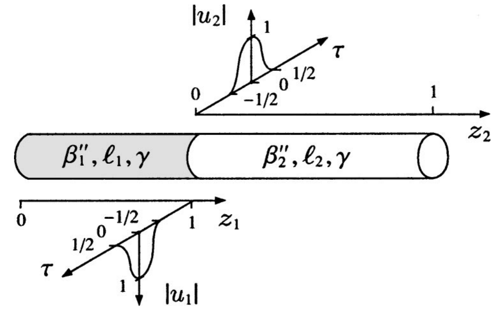

FIG. 1. Boundary scaling in a two-step dispersion map.

$$\beta_j\equiv\frac{\beta_j''\ell_j}{T_B^2}, \tag{21}$$

$$\kappa_j\equiv\gamma A_B^2\ell_j. \tag{22}$$

This choice of scaling is depicted in Fig. 1.

The propagation through one map period can then be considered as a sequential process, where the field is evolved according to two separate NLS systems, continuity being explicitly enforced at the boundary

$$u_1(1,\tau)=u_2(0,\tau)\equiv u_B(\tau). \tag{23}$$

In order to describe quasiperiodic solutions to this system, it must be required that the field acquire at most a constant phase ϕ after having propagated through one map period. One way to achieve this is to impose a quasiconjugate propagation scheme in each fiber [8]

$$u_j(1,\tau)=u_j^*(0,\tau)e^{i\theta_j}. \tag{24}$$

When coupled to the boundary condition (23), this gives rise to quasiperiodic behavior with a phase shift $\phi=\theta_2-\theta_1$. According to Eq. (15), the scattering data at the fiber extremities are then simply related by

$$\lambda_{k,j}(1)=-\lambda_{k,j}^*(0), \tag{25a}$$

$$b_j(1,\lambda)=-e^{i\theta_j}b_j^*(0,-\lambda^*). \tag{25b}$$

But since Eq. (19a) implies the discrete eigenvalues to be constant through z_j, half of these constraints reduce to $\lambda_{k,j}^*=-\lambda_{k,j}$, which is simply the discrete spectrum parity condition (12a). The condition (25a) may therefore be satisfied simply by requiring the field to have a definite parity. According to Eq. (12a), the discrete eigenvalues may thus be parametrized as $\lambda_{k,j}=i\mu_{k,j}$, with $\mu_{k,j}\in\mathbb{R}^+$. Following Ref. [10], the parabolic evolution (19b) of $b_j(z_j,\lambda)$ from $z_j=0$ to $z_j=1$ may then be combined with the quasiconjugate condition (25b) and the parity condition (12b) to yield

$$b_j(0,\lambda)=-\mathcal{P}b_j^*(0,\lambda)e^{-i(2\beta_j\lambda^2-\theta_j)}. \tag{26}$$

Introducing the complex parametrization

$$b_j(0,\lambda)\equiv f_j(\lambda)e^{i\omega_j(\lambda)} \tag{27}$$

with $\mathrm{Im}[f_j(\lambda)]=\mathrm{Im}[\omega_j(\lambda)]=0$, and using the fact that $\lambda^2 \in \mathbb{R}$ even for the discrete points $\lambda_{k,j}=i\mu_{k,j}$ gives

$$f_j(\lambda)=\mathcal{P}f_j(-\lambda), \tag{28}$$

$$\omega_j(\lambda)=-\beta_j\lambda^2+\frac{\theta_j}{2}+\frac{\pi}{4}(\mathcal{P}+1) \quad (\mathrm{mod}\,\pi). \tag{29}$$

Absorbing the modπ factor in the unknown $f_j(\lambda)$, $b_j(z_j,\lambda)$ is seen to be parabolically chirped according to

$$b_j(z_j,\lambda)=f_j(\lambda)e^{i\{\beta_j(2z_j-1)\lambda^2+\theta_j/2+(\pi/4)(\mathcal{P}+1)\}}. \tag{30}$$

B. Direct scattering perturbation scheme

The parametrization (30), together with the parity assumption, translates all the information about the quasiconjugate propagation scheme (24) onto the scattering data, and the problem has now been reduced to finding which phases θ_j and moduli $f_j(\lambda)$ also satisfy the boundary condition (23). It will now be shown that it is possible to treat this problem in perturbation theory. The direct scattering transform is well suited to such a task because infinitely iterating the integral scattering problem (16b) and using Eq. (17b) yields the power series

$$b_j(z_j,\lambda)=\sum_{n=1}^{\infty} i^{2n-1}\sigma_j^{n-1}b_j^{(2n-1)}(z_j,2\lambda)\alpha_j^{2n-1}, \tag{31}$$

where the following functionals of u have been defined:

$$b_j^{(1)}(z_j,\lambda)\equiv\int_{-\infty}^{\infty}d\tau e^{-i\lambda\tau}u_j(z_j,\tau), \tag{32a}$$

$$\begin{aligned}b_j^{(2n-1)}(z_j,\lambda)\equiv&\prod_{r=1}^{2n-1}\left\{\int_{-\infty}^{\infty}d\tau_s e^{i(-)^r\lambda\tau_r}\right\}\\&\times\prod_{\ell=1}^{2n-2}\{\theta(\tau_\ell-\tau_{\ell+1})\}\prod_{m=1}^{n}\{u_j(z_j,\tau_{2m-1})\}\\&\times\prod_{s=1}^{n-1}\{u_j^*(z_j,\tau_{2s})\},\end{aligned} \tag{32b}$$

and where $\theta(x-y)$ is the unit step function

$$\theta(x-y)\equiv\begin{cases}0 & \text{if} \quad x<y,\\ 1 & \text{if} \quad x>y.\end{cases} \tag{33}$$

On the other hand, perturbation expansions for the modulus and phase of $b_j(z_j,\lambda)$ may be introduced in the form

$$f_j(\lambda)=\sum_{n=0}^{\infty} f_j^{(n)}(\lambda)\alpha_j^n, \tag{34a}$$

$$\theta_j=\sum_{n=0}^{\infty}\theta_j^{(n)}\alpha_j^n, \tag{34b}$$

where the coefficients $f_j^{(n)}(\lambda)$ and $\theta_j^{(n)}$ are real. Substituting these expansions in Eq. (30) then yields

$$b_j(z_j,\lambda)=e^{i\{\beta_j(2z_j-1)\lambda^2+(1/2)\theta_j^{(0)}+(\pi/4)(\mathcal{P}+1)\}}\sum_{n=0}^{\infty}c_j^{(n)}(k)\alpha_j^n, \tag{35}$$

where the expansion coefficients are given by

$$\begin{aligned}c_j^{(p)}(\lambda)\equiv& f_j^{(p)}(\lambda)\\&+\sum_{\substack{m=0\\p>1}}^{p-1}f_j^{(m)}(\lambda)\sum_{n=1}^{\infty}\frac{i^n}{2^n n!}\sum_{\substack{r_1\ldots r_n=1\\ \sum_{\ell=1}^{n}r_\ell+m=p}}^{\infty}\prod_{\ell=1}^{n}\theta_j^{(r_\ell)}.\end{aligned} \tag{36}$$

The proposed approach is to consider an unspecified field $\hat{u}_B(k)$ at the boundary and use Eq. (31) to construct its scattering data in each fiber, up to order N in α_j. Imposing these data to be of the quasiconjugate form (35) order by order will then yield the conditions for $\hat{u}_B(k)$ to exhibit Nth order quasiperiodic behavior around $\alpha_j=0$.

This program is not too difficult to realize for $\lambda=k\in\mathbb{R}$, but there is a subtlety associated with the discrete spectrum because the eigenvalues $\mu_{k,j}$ depend on α_j, and this implicit dependence should also be considered before the expansion (31) is evaluated on the imaginary axis and employed in any perturbation scheme. Unfortunately, this information is difficult to incorporate since there is no simple way of obtaining an expansion for $\mu_{k,j}(\alpha_j)$ without the exact analytical knowledge of the boundary field. This problem will be sidestepped by simply requiring the field to be purely radiative, i.e., it is required that $\Lambda_j=\varnothing$. This is not as restrictive as it may seem because the scaling depicted in Fig. 1 implies the coefficients $b_j^{(2n-1)}(z_j,2\lambda)$ to be roughly of order one at the boundary and so the validity of any low-order truncation of Eq. (31) requires $\alpha_j\ll 1$, which is equivalent to

$$|\beta_j''|\gg|\gamma|A_B^2T_B^2. \tag{37}$$

On the other hand, there can be no discrete spectrum in any normal fiber and it can be shown [12] that a necessary and sufficient condition for the absence of a discrete spectrum in any anomalous fiber is given by

$$\int_{-\infty}^{\infty}|u_j(z_j,\tau)|d\tau<\frac{\ln(2+\sqrt{3})}{\alpha_j}. \tag{38}$$

Now our choice of scaling also implies this quadrature to be of order 1 at the boundary and if α_j is small, the bound becomes large so that any field profile will be purely radiative in the perturbation regime (37). Restricting the perturbative analysis to the real axis, the Fourier-transformed field

$$\hat{u}_j(z_j,k) \equiv \int_{-\infty}^{\infty} d\tau e^{-ik\tau} u_j(z_j,\tau), \tag{39a}$$

and the integral representation for the unit step function

$$\theta(x-y) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \int_0^{\infty} ds e^{ip(x-y-s)}, \tag{40}$$

may be invoked to reduce the scattering coefficients (32) to the form

$$b_j^{(1)}(z_j,k) = \hat{u}_j(z_j,k), \tag{41a}$$

$$\begin{aligned} b_j^{(2n-1)}(z_j,k) = &\prod_{r=1}^{2n-2} \left\{ \int_0^{\infty} d\tau_r \int_{-\infty}^{\infty} \frac{dp_r}{2\pi} e^{i(-1)^r \tau_r p_r} \right\} \\ &\times \hat{u}_j(z_j,p_1+k) \prod_{\ell=1}^{n-1} \{\hat{u}_j^*(z_j,p_{2\ell-1}+p_{2\ell}+k)\} \\ &\times \prod_{\substack{m=1\\ n>2}}^{n-2} \{\hat{u}_j(z_j,p_{2m}+p_{2m+1}+k)\} \\ &\times \hat{u}_j(z_j,p_{2n-2}+k). \end{aligned} \tag{41b}$$

Equating the quasiconjugate condition (35) with the scattering expansion (31) at the junction then gives

$$\begin{aligned} &\sum_{n=1}^{\infty} i^{2n-1} \sigma_j^{n-1} b_B^{(2n-1)}(2k) \alpha_j^{2n-1} \\ &= e^{i\{(-1)^{j-1}\beta_j k^2 + (1/2)\theta_j^{(0)} + (\pi/4)(\mathcal{P}+1)\}} \sum_{n=0}^{\infty} c_j^{(n)}(k) \alpha_j^n, \end{aligned} \tag{42}$$

where the boundary coefficients are defined by

$$b_B^{(2n-1)}(2k) \equiv b_1^{(2n-1)}(1,2k) = b_2^{(2n-1)}(0,2k). \tag{43}$$

C. Low-order perturbative analysis

Equation (42) is the starting point for the proposed perturbative analysis. At order α_j^0, it implies $f_1^{(0)}(k) = f_2^{(0)}(k) = 0$. This result seems to imply that the only possibility for linear quasiperiodicity is the trivial solution $\hat{u}_B(k)=0$, but it is well known that any linear pulse profile is exactly periodic for a zero-average dispersion map. This discrepancy is due to the fact that the scattering formalism is strictly not related to the NLS at $\alpha_j=0$. Indeed, in order to obtain the NLS equation from the compatibility condition (6), one has to divide the latter by α so the correspondence breaks down at the linear point and the zeroth-order constraint is actually not related to linear quasiperiodicity. Proceeding then to first order, the conditions take the form

$$i b_B^{(1)}(2k) = f_j^{(1)}(k) e^{i\{(-1)^{j-1}\beta_j k^2 + \theta_j^{(0)}/2 + (\pi/4)(\mathcal{P}+1)\}}, \tag{44}$$

which are compatible only if

$$\beta_1 = -\beta_2 \equiv \beta, \tag{45a}$$

$$\theta_1^{(0)} = \theta_2^{(1)} \equiv \theta^{(0)}, \tag{45b}$$

$$f_1^{(1)}(k) = f_2^{(1)} \equiv f^{(1)}(k). \tag{45c}$$

Now the direct scattering problem shows that $\theta^{(0)}$ may be arbitrarily shifted by simply gauging the global phase of $\hat{u}_B(k)$ so fixing this gauge according to $\theta^{(0)} = -\pi/2(\mathcal{P}-1)$, the specific form of the scattering transform coefficients (41a) implies the junction field to be given by

$$\hat{u}_B(k) = f^{(1)}\left(\frac{k}{2}\right) e^{i(\beta/4)k^2}. \tag{46}$$

Compared to the linear case, first-order quasiperiodicity around $\alpha_j=0$ still requires the average dispersion to vanish and still allows for an infinity of solutions, but it now singles out those junction profiles which have a specific parabolic chirp in Fourier-space. The result (46) may then be used to explicitly separate the scattering coefficients into real and imaginary parts according to

$$\begin{aligned} \mathrm{Re}\{b_B^{(2n-1)}(2k)\} = &\cos(\beta\ell k^2) I_c^{(2n-1)}(k) \\ &+ \sin(\beta\ell k^2) I_s^{(2n-1)}(k), \end{aligned} \tag{47a}$$

$$\begin{aligned} \mathrm{Im}\{b_B^{(2n-1)}(2k)\} = &\sin(\beta\ell k^2) I_c^{(2n-1)}(k) \\ &- \cos(\beta\ell k^2) I_s^{(2n-1)}(k), \end{aligned} \tag{47b}$$

with the introduction of the quantities [18] $I_s^{(1)}(k) \equiv 0$,

$$\begin{aligned} I_s^{(2n-1)}(k) \equiv &2^{2n-2} \prod_{r=1}^{2n-2} \left\{ \int_0^{\infty} d\tau_r \int_{-\infty}^{\infty} \frac{dp_r}{2\pi} e^{i(-1)^r \tau_r p_r} \right\} \\ &\times \sin\left(\sum_{s=1}^{2n-3} (-1)^{s+1} 2\beta\ell p_s p_{s+1} \right) f^{(1)}(p_1+k) \\ &\times \prod_{\ell=1}^{2n-3} \{f^{(1)}(p_\ell+p_{\ell+1}+k)\} f^{(1)}(p_{2n-2}+k), \end{aligned} \tag{48a}$$

and $I_c^{(1)}(k) \equiv f^{(1)}(k)$,

$$\begin{aligned} I_c^{(2n-1)}(k) \equiv &2^{2n-2} \prod_{r=1}^{2n-2} \left\{ \int_0^{\infty} d\tau_r \int_{-\infty}^{\infty} \frac{dp_r}{2\pi} e^{i(-1)^r \tau_r p_r} \right\} \\ &\times \cos\left(\sum_{s=1}^{2n-3} (-1)^{s+1} 2\beta\ell p_s p_{s+1} \right) f^{(1)}(p_1+k) \\ &\times \prod_{\ell=1}^{2n-3} \{f^{(1)}(p_\ell+p_{\ell+1}+k)\} f^{(1)}(p_{2n-2}+k). \end{aligned} \tag{48b}$$

Proceeding now to second order, the fact that $\theta_j^{(n)}$ and $f_j^{(n)}(k)$ are real can be invoked to obtain $f_j^{(2)}(k)=\theta_j^{(1)}=0$. Using this second-order result, the third-order equations may be written as

$$-ib_B^{(3)}(2k)=\sigma_j\left(f_j^{(3)}(k)+\frac{i}{2}\theta_j^{(2)}f^{(1)}(k)\right)ie^{i\beta k^2}, \quad (49)$$

and subtracting these in both fibers gives

$$\sigma_1 f_1^{(3)}(k)=\sigma_2 f_2^{(3)}(k)\equiv f^{(3)}(k), \quad (50a)$$

$$\sigma_1\theta_1^{(2)}=\sigma_2\theta_2^{(2)}\equiv\theta^{(2)}. \quad (50b)$$

Separating the third-order system (49) into real and imaginary parts and substituting the results (47a) and (47b) then gives

$$[f^{(3)}(k)+I_c^{(3)}(k)]\sin(\beta\ell k^2)=\left(I_s^{(3)}(k)-\frac{\theta^{(2)}}{2}f^{(1)}(k)\right)\times\cos(\beta\ell k^2), \quad (51a)$$

$$[f^{(3)}(k)+I_c^{(3)}(k)]\cos(\beta\ell k^2)=\left(\frac{\theta^{(2)}}{2}f^{(1)}(k)-I_s^{(3)}(k)\right)\times\sin(\beta\ell k^2), \quad (51b)$$

which are compatible only if

$$f^{(3)}(k)=-I_c^{(3)}(k), \quad (52)$$

$$I_s^{(3)}(k)=\frac{\theta^{(2)}}{2}f^{(1)}(k). \quad (53)$$

The second of these expressions represents a nontrivial constraint for the determination of the junction-field modulus $f^{(1)}(k/2)$ and the second-order phase shift $\phi^{(2)}\equiv\theta_2^{(2)}-\theta_1^{(2)}$. Indeed, since the nonlinearity in both fibers is the same but the dispersions alternate in sign [see Eq. (45a)], one has $\sigma_1=-\sigma_2$ so that $\theta^{(2)}$ is actually given by $\theta^{(2)}=(\sigma_2/2)\phi^{(2)}$. In order to simplify this constraint, one may write $I_s^{(3)}(k)$ in the form

$$I_s^{(3)}(k)=4\int_0^\infty ds\int_0^\infty dt\int_{-\infty}^\infty\frac{dp}{2\pi}e^{-2itp}G(s,k,p), \quad (54)$$

where

$$G(s,k,p)\equiv\int_{-\infty}^\infty\frac{dq}{2\pi}e^{2isq}H(k,p,q), \quad (55)$$

$$H(k,p,q)\equiv\sin(2\beta pq)f^{(1)}(k+p)f^{(1)}(k+p+q)f^{(1)}(q+k). \quad (56)$$

The integral over p can then obviously be bounded by

$$\left|\int_{-\infty}^\infty\frac{dp}{2\pi}e^{-2itp}G(s,k,p)\right|\leqslant\int_{-\infty}^\infty\frac{|dp||dq|}{(2\pi)^2}|f^{(1)}(k+p)|\times|f^{(1)}(k+p+q)||f^{(1)}(q+k)|. \quad (57)$$

Now energy conservation in the anomalous fiber implies $|b(k)|\leqslant 1$ so $f^{(1)}(k)$ is known to be bounded. Assuming also $u_B(\tau)$ to vanish faster than any algebraic function implies $f^{(1)}(k)$ to be analytic on the real axis, and the right-hand side of Eq. (57) is thus seen to converge to some finite value. Since this bound is also independent of s and t, the left-hand side is seen to be uniformly convergent with respect to these variables. Interchanging then the t and p integrals and integrating $I_s^{(3)}(k)$ over t gives

$$I_s^{(3)}(k)=4\int_0^\infty ds\int_{-\infty}^\infty\frac{dp}{2\pi}\left\{\lim_{t\to\infty}e^{-2itp}\frac{G(s,k,p)}{-2ip}-\frac{G(s,k,p)}{-2ip}\right\}. \quad (58)$$

Now since $|\sin(2\beta qp)/p|$ can be bounded by $|2\beta q|$, one has

$$\left|\int_{-\infty}^\infty\frac{dp}{2\pi}e^{-2itp}\frac{G(s,k,p)}{2ip}\right|\leqslant 2|\beta|\int_{-\infty}^\infty\frac{|dp||dq|}{(2\pi)^2}|q|\times|f^{(1)}(k+p)||f^{(1)}(k+p+q)|\times|f^{(1)}(q+k)|. \quad (59)$$

If $f^{(1)}(k)$ is now assumed to decay fast enough as $|k|\to\infty$, so that the right-hand side of Eq. (59) is finite, its left-hand side becomes uniformly convergent with respect to s and t. It is thus possible to take the limit in the first term of Eq. (58) outside the p integral and this term becomes proportional to a Fourier transform evaluated at infinity

$$\lim_{t\to\infty}\int_{-\infty}^\infty dp\,e^{-i(2t)p}\frac{G(s,k,p)}{p}, \quad (60)$$

where $G(s,k,p)/p$ has already been shown to be integrable by Eq. (59). Now the Riemann-Lebesgue lemma states that the Fourier transform of any Riemann-integrable function vanishes at $\pm\infty$. This limit may thus be eliminated and there remains

$$I_s^{(3)}(k)=4\int_0^\infty ds\int_{-\infty}^\infty\frac{dp}{2\pi}\int_{-\infty}^\infty\frac{dq}{2\pi}e^{2isq}\frac{H(k,p,q)}{2ip}. \quad (61)$$

A reasoning similar to that above then allows one to interchange the s and q integrals, integrate over s and use once more the Riemann-Lebesgue lemma to yield

$$I_s^{(3)}(k)=\int_{-\infty}^\infty\frac{dp\,dq}{(2\pi)^2}\frac{\sin(2\beta pq)}{pq}f^{(1)}(k+p)\times f^{(1)}(k+p+q)f^{(1)}(q+k). \quad (62)$$

Using this simplification [19], the integral constraint (53) may finally be written as

$$\frac{\sigma_2}{4}\phi^{(2)}f^{(1)}(k)=\int_{-\infty}^{\infty}\frac{dp\,dq}{(2\pi)^2}\,\frac{\sin(2\beta pq)}{pq}\,f^{(1)}(k+p)\times f^{(1)}(k+p+q)f^{(1)}(q+k). \qquad (63)$$

This is exactly the quasiperiodicity condition arising from the zero-average dispersion DMNLS [8,9,14] and its numerics have already been shown to support even and odd solutions, whose power spectra exhibit respectively one and two exponential humps [9,15].

It should be emphasized that in other perturbative models, the physical interpretation of the DMNLS is not always as clear as in the present context. For example, in the regular perturbative approach [8], the unknown function satisfying the DMNLS is found to be the field modulus at the fiber midpoint $|\hat{u}_1(1/2,k)|$, which may only be identified with $f^{(1)}(k/2)$ if the field is assumed to propagate at order 1 in α_j between the fiber midpoint and the junction. This is clearly less precise than the present α_j^3 treatment. The multiscale analysis also lacks some precision, since its solution to the DMNLS represents the asymptotic field envelope for a slow z scale. In contrast, by lifting small scale assumptions on the fiber lengths and propagating the field exactly, the present procedure can identify the solution to the DMNLS very precisely as the field modulus at the junction.

D. Higher-order perturbative analysis

Ablowitz *et al.* have also succeeded in deriving the higher-order DMNLS (HODMNLS), a higher-order correction to Eq. (63) that describes DM solitons with a large number of humps [16]. In order to compare the present formulation with this correction, let us proceed to order 4, where it is seen that $f_j^{(4)}(k)=\theta_j^{(3)}=0$. This and the previous lower-order results then allow one to write the fifth-order constraints

$$b_B^{(5)}(2k)=\left\{f_j^{(5)}(k)+\frac{i}{2}\theta_j^{(2)}f_j^{(3)}(k)+\left(\frac{i}{2}\theta_j^{(4)}-\frac{\theta_j^{(2)}\theta_j^{(2)}}{8}\right)f^{(1)}(k)\right\}e^{i\beta k^2}. \qquad (64)$$

Substituting now $f_j^{(3)}(k)=-\sigma_j I_c^{(3)}(k)$, $\theta_j^{(2)}=\sigma_j\theta^{(2)}$ and subtracting these equations in both fibers gives

$$f_1^{(5)}(k)=f_2^{(5)}(k)\equiv f^{(5)}(k), \qquad (65a)$$

$$\theta_1^{(4)}=\theta_2^{(4)}\equiv\theta^{(4)}. \qquad (65b)$$

In terms of these quantities and Eqs. (47a) and (47b), the fifth-order system separates into real and imaginary parts according to

$$\cos(\beta k^2)\left(f^{(5)}(k)-\frac{\theta^{(2)}\theta^{(2)}}{8}f^{(1)}(k)-I_c^{(5)}(k)\right)=\sin(\beta k^2)\left(I_s^{(5)}(k)-\frac{\theta^{(2)}}{2}I_c^{(3)}(k)+\frac{\theta^{(4)}}{2}f^{(1)}(k)\right), \qquad (66a)$$

$$\sin(\beta k^2)\left(f^{(5)}(k)-\frac{\theta^{(2)}\theta^{(2)}}{8}f^{(1)}(k)-I_c^{(5)}(k)\right)=-\cos(\beta k^2)\left(I_s^{(5)}(k)-\frac{\theta^{(2)}}{2}I_c^{(3)}(k)+\frac{\theta^{(4)}}{2}f^{(1)}(k)\right), \qquad (66b)$$

which are compatible provided that

$$f^{(5)}(k)=I_c^{(5)}(k)+\frac{\theta^{(2)}\theta^{(2)}}{8}f^{(1)}(k), \qquad (67)$$

$$I_s^{(5)}(k)-\frac{\theta^{(2)}}{2}I_c^{(3)}(k)=-\frac{\theta^{(4)}}{2}f^{(1)}(k). \qquad (68)$$

It is also trivial to see that at order 6, $f_j^{(6)}(k)=\theta_j^{(5)}=0$ so that any function $f^{(1)}(k)$ satisfying simultaneously Eqs. (63) and (68) would represent a solution that is quasiperiodic up to order 6. However, since Eq. (68) does not have the form of a correction but represents a whole new constraint, the existence of such high-order solutions is rather unlikely. Note also that higher-order DM solitons obtained through the multiple scales analysis not only depend on the scaled dispersion β, but also on the explicit ratio of ℓ_1 to ℓ_2. Since Eq. (68) only depends on the dispersion, it is necessarily different from the HODMNLS and therefore does not describe higher-order solitons. This lack of concordance at high orders is to be expected since the two approaches are not equivalent, the multiple scales analysis requiring a robustness of its solutions with respect to perturbations of the small map period $\ell_1+\ell_2$, whereas the present perturbation parameter is α_j.

IV. CONCLUSION

An approach to the study of dispersion management has been presented, where the field is exactly propagated and the quasiperiodicity property is approximated in perturbation theory. This approach actually hunts down those solutions to the linear DM problem which remain quasiperiodic when the nonlinearity is turned on and varied up to order 4. These robust quasiperiodic fields have been shown to correspond with the solutions of the lowest-order DMNLS, to be purely radiative, and to exist only for zero-average dispersion maps. However, it is well known that quasiperiodic fields also exist in the case of nonzero residual dispersion [17]. These solutions were excluded from the present formalism because they are simply not valid in any neighborhood of $\alpha_j = 0$. In order to treat these cases, the perturbation procedure should be modified so that quasiperiodicity conditions are not imposed at each order but only at some finite point α_j, or in the neighborhood of such a point that excludes $\alpha_j = 0$. Such a modification should also provide the link with Ablowitz' HODMNLS correction [16].

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[18] Note that $I_s^{(2n-1)}(k)$ and $I_c^{(2n-1)}(k)$ can be shown to be real by complex conjugating and changing variables according to $\tau_j \leftrightarrow \tau_{2n-1-j}$, $p_j \leftrightarrow p_{2n-1-j}$.
[19] Note that the Riemann-Lebesgue lemma cannot be used to simplify $I_c^{(3)}(k)$ since in that case, integrating the exponentials produces the limit of a nonintegrable kernel.